SCHRÖDINGER WAVE FUNCTION FOR A FREE FALLING PARTICLE IN THE SCHWARZSCHILD BLACK HOLE

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Abstract

We use the time-dependent invariant method in a geometric approach (Jacobi fields) to quantize the motion of a free falling point particle in the Schwarzschild black hole. Assuming that the particle comes from infinity, we obtain the relativistic Schrödinger wave function for this system.

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1   Introduction

The study of the quantization of the motion of particles in a general gravitational background space-time has attracted considerable interest in the literature[1]. These analyses offer the opportunity to be a good theoretical laboratories in the attempt to construct an effective quantum theory of gravitation.

The General Relativity Theory assures us that every coordinate frame is physically equivalent from a classical point of view. Thus we can affirm that any falling particle in a gravitational field will be accelerated relatively to any stationary coordinate frame defined by a Killing vector. However, it is possible that a specific space-time has no Killing vector, so in this case we would not have at our disposal a frame which enables us to measure this acceleration. In such a situation, the best we can do is to take two different particles and measure their relative acceleration. This relative accelaration will be given by the equation which governs the Jacobi fields[2]. These equations, which are classical, can indicates us an analogous quantum mechanical system which permits us to study the corresponding quantum motion of the particle in this general gravitational background space-time. This is what happers for a free falling particle in the Schwarzschild space-time. As we shall see, the classical Jacobi field equations for a free falling particle in the Schwarzschild metric space-time is similar to a classical harmonic ociolator with a time-dependent frequence. Fortunately there exists a well defined procedure to quantize systems like that through the use of invariants[3, 4]. This technique has been demonstrated to be a powerful one to quantize such systems, and to yeld a closed expression for the Schrödinger wave function.

In the present paper we shall use the theory of explicitly time-dependent invariants to quantize the free motion of a point particle in the Schwarzschild black hole using Jacobi fields. The time parameter in the relativistic Schrödinger equation will be the affine parameter, the proper time of the particle in this manifold. As will become clear, this geometric approach has two advantages: it avoids the ordering problem of differential operators in a general background space-time[5], and it also avoids the horizon problem in our specific space-time, because, as we shall see, our relativistic wave function is well defined at $r = 2M$.

So the main objective of this paper is to study from quantum point of

view the motion of a free falling particle in the Schwarzschild black hole.

This paper is organized as follows. In Sec. 2 we introduce the Jacobi field equations for general case. In Sec. 3 the invariant method and the relativistic Schrödinger equation associated with this geometric approach are analized. In Sec. 4 we apply the formalism reviewed in the Sec. 3 to study the motion of a free point particle falling in the Schwarzschild black hole. In Sec. 5 we summarize the main results of this work.

2   Jacobi Fields

In this section we briefly review the Jacobi fields and their respective differential equation for a general manifold[6].

Let us consider a differentiable manifold, $\mathcal{M}$, and two structures defined on $\mathcal{M}$, namely affine connection, ∇ , and Riemann tensor, R , related by the generic equation

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad (2.1)$$

where X, Y and Z are vector fields in the tangent space.

The torsion tensor can be defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \qquad (2.2)$$

where $\nabla_X Y$ is the covariant derivative of the field Y along the X direction. We also can define the Lie derivative by

$$\mathcal{L}_X Y = [X,Y]. \qquad (2.3)$$

For a connection free of torsion we can write

$$\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X. \qquad (2.4)$$

On the other hand, we are interested in constructing a family of curves $\lambda(t,s)$, moving each point $\lambda(t)$ a distance s along the integrals curves of $V = (\frac{\partial}{\partial t})_{\lambda(t,s)}$. Let the vector field Z be equal to $(\frac{\partial}{\partial t})_{\lambda(t,s)}$, so that we have a family of curves with the condition

$$\mathcal{L}_V Z = 0. \qquad (2.5)$$

From (2.5), we can easily see that the equations which govern the Jacobi field can be written in the form

$$\nabla_V \nabla_V Z + R(..., V, Z, V) - \nabla_Z \nabla_V V = 0. \tag{2.6}$$

In this paper we are interested in studying the quantum motion of a point particle interacting with a background gravitational field, so, the particle's world lines are geodesics in this space-time, whose equations are given by

$$\nabla_V V = 0. \tag{2.7}$$

In this case the Jacobi equations are reduced to a geodesic desviation, assuming a simpler form, and the Fermi derivative $\frac{D_F Z}{\partial s}$ coincides with the usual covariant derivative[7]:

$$\frac{D_F Z}{\partial s} = \frac{DZ}{\partial s} = \nabla_V Z.$$

Because the particle possesses a non vanishing rest mass, it is convenient to define the tangent vector V as a time-like one, so that $g(V, V) = -1$. Considering e_o, e_1, e_2, and e_3 be an orthogonal basis of the space-time, in the rest frame of the particle, $V = e_o$. So, taking the Fermi-Walker transport into Eq. (2.6), we can write

$$\frac{d^2 Z_i}{d\tau^2} + R_{0i0j} Z_j = 0, \tag{2.8}$$

where τ is, in general, an affine parameter, which in our case is the proper time of the particle and Z_i the component of the space-like vector Z, with $g(Z, V) = 0$. We can observe that the Fermi-Walker transport for our case, maps the stationary Schwarzschild coordinate system into another one, contructed by successive Lorentz boosts, where the particle, whose motion is governed by (2.8), interacts with the curvature of the space-time as an external force. The system constructed by the successive Lorentz frames is not conservative, so at quantum level, the probability density is explicitly time-dependent as we one can see from the wave function given in Sec.4.

For the case where R_{0i0j} is diagonal in the spatial indices [8], the Eq. (2.8) becomes

$$\frac{d^2 Z_i}{d\tau^2} + R_{0i0i} Z_i = 0, \tag{2.9}$$

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with no summation indices assumed.

As we have already mentioned, in this paper we shall specialize to the Schwarzschild black-hole case. A more general analysis is now under investigation.

3   Time-Dependent Invariants and the Relativistic Schrödinger Equation

In this section we shall briefly review some important points related with the invariant method which we shall employ to quantize the motion of a free falling particle in a specific space-time through a geometric approach.

Let as now consider the following time-dependent Hamiltonian

$$H(\tau) = \sum_{j=1}^{3} \left[\frac{P_j^2}{2m} + \frac{m}{2} R_{0i0i}(\tau) Z_j^2 \right], \tag{3.1}$$

where P_i is the canonical momentum conjugated to the coodinate Z_i. We can notice that R_{0i0i} is not a variable on phase space, and that the equation above is formally similar to an anisotropic time-dependent harmonic oscilator, whose classical equation for any direction is given by Eq.(2.9).

In order to quantize this system we shall employ the invariant method which depends on the so called invariant operator. Fo our system this operator is expressed by [4, 9, 10]

$$I(\tau) = \frac{1}{2} \sum_{i=1}^{3} \left[K_i \left(\frac{Z_i}{\sigma_i} \right)^2 + (\sigma_i P_i - m \dot{\sigma}_i Z_i)^2 \right], \tag{3.2}$$

where K_i is an arbitrary constant and $\sigma_i(\tau)$ a c-number obeying the inhomogeneous auxiliary equation

$$\frac{d^2 \sigma_i}{d\tau^2} + R_{0i0i}(\tau) \sigma_i = \frac{K_i}{m^2 \sigma_i^3}. \tag{3.3}$$

The invariant operator $I(\tau)$ satisfies the equation [3, 4]

$$\frac{dI(\tau)}{d\tau} = \frac{\partial I(\tau)}{\partial \tau} + \frac{1}{i\hbar} [I, H] = 0. \tag{3.4}$$

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Because the operators $I(\tau)$ and $H(\tau)$ are written as the sum of three independent expressions in each direction, the solution of the equation of motion can be written as a product of solutions, each refering to one direction. So we shall focus our attention just to one direction. Let $\phi_{\lambda_i}(Z_i, \tau)$ be the eigenfunction of the operator $I_i(\tau)$ with eigenvalue λ_i. Thus we have:

$$I_i(\tau)\phi_{\lambda_i}(Z_i, \tau) = \lambda_i \phi_{\lambda_i}. \tag{3.5}$$

Next let us consider the Schrödinger equation

$$i\hbar \frac{\partial}{\partial \tau} \Psi_{\lambda_i}(Z_i, \tau) = H_i(\tau)\Psi_{\lambda_i}(Z_i, \tau), \tag{3.6}$$

with

$$H_i(\tau) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial Z_i^2} + \frac{1}{2}mR_{0i0i}(\tau)Z_i^2, \tag{3.7}$$

where we have used the operator P_i in its differential form.

In order to obtain the solution for the relativistic Schrödinger equation[1] (3.6), we shall use the Risenfeld an Lewis (RL) method [3, 4], which permits to express Ψ_{λ_i} in terms of another function ϕ_{λ_i} by

$$\Psi_{\lambda_i}(Z_i, \tau) = \exp[i\alpha_{\lambda_i}(\tau)]\phi_{\lambda_i}(Z_i, \tau), \tag{3.8}$$

$\alpha_{\lambda_i}(\tau)$ being the phase factor which obeys the equation

$$< \phi_{\lambda_i} \mid \phi_{\lambda_j} > \hbar \frac{\partial \alpha_{\lambda_j}}{\partial \tau} =< \phi_{\lambda_i} \mid i\hbar\frac{\partial}{\partial \tau} - H(\tau) \mid \phi_{\lambda_j} > . \tag{3.9}$$

The new function ϕ_{λ_i}, on the other hand, is obtained, by the unitary transformation,

$$\phi'_{\lambda_i} = U_i \phi_{\lambda_i} = \exp\left[-\frac{im}{2\hbar}\frac{\dot{\sigma}}{\sigma}Z_i\right]\phi_{\lambda_i} \tag{3.10}$$

with the solution of the eigen-values equation

$$I'_i\phi'_{\lambda_i} = \lambda_i \phi'_{\lambda_i} \tag{3.11}$$

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[1]Although Eq.(3.6) is not in a covariant form, it is in fact derived from the relativistic time-dependent Hamiltonian Eq.(3.1)

where I_i' is operator

$$I_i' = -\frac{\hbar^2}{2}\sigma_i^2\frac{\partial^2}{\partial Z_i^2} + \frac{K_i}{2}\frac{Z_i^2}{\sigma_i^2} \qquad , \qquad (3.12)$$

similar to the harmonic oscilator Hamiltonian.

Again, defining a new variable $X_i = \frac{Z_i}{\sigma_i}$ and the field $\phi_{\lambda_i} = \sigma_i^{1/2}\phi_{\lambda_i}'$, we get the eigenvalue equation

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dX_i^2} - \frac{K_i}{2}X_i^2\right)\varphi_{\lambda_i} = \lambda_i\varphi_{\lambda_i}. \qquad (3.13)$$

Thus the objective of the invariant method (RL) is to express a general solution of a time-dependent Schrödinger equation in terms of some well known eigenvalue differential equation. For our specific case, this was an harmonic oscilator differential equation. The only problem now is to solve the inhomogeneous auxiliary equation (3.4). Fortunatly, although this equation is not linear, we were able to obtain, for the problem that we shall analyse in the next section, different solutions for $\sigma_i(\tau)$ for positive and negative values of the constant K_i, which, on the other hand, leads us to an attractive and repulsive, respectively, harmonic potential into (3.13).

4 The Relativistic Schrödinger Wave Function for a Particle in the Schwarzschild Black Hole

The exterior line element of the Schwarzschild black-hole is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \qquad (4.1)$$

In accordance with Eq.(2.9), the non-vanishing Riemann tensor components of interest in this space-time are:

$$R_{0101} = -\frac{2M}{r^3} \qquad , \qquad (4.2)$$

and

$$R_{0202} = R_{0303} = \frac{M}{r^3}. \tag{4.3}$$

where the above relations are invariant under a boost along the radial direction for r> 2M, and still remain valid for r≤ 2M [11].

As we have pointed out at the end of the last section, in order to obtain the Schrödinger wave function $\Psi_{\lambda_i}(Z_i, \tau)$, we have first to solve the auxiliary equation (3.4), which, on the other hands, depends on the function $R_{0i0i}(\tau)$. This function can be constructed through the differential equations obeyed by the classical Jacobi fields, which for this geometry read

$$\frac{d^2 Z_i}{d\tau^2} - \frac{2M}{r^3} Z_i = 0, \tag{4.4}$$

and

$$\frac{d^2 Z_j}{d\tau^2} + \frac{M}{r^3} Z_j = 0, \tag{4.5}$$

where the index j assumes values 2 e 3 in the last equation. Now assuming that the test particle is at rest at infinity, it is possible to show [12] the relationship between the radial coordinate and the proper time

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \tag{4.6}$$

whose solutions is

$$\tau = -\frac{2}{3\sqrt{2M}} r^{3/2} + const. \tag{4.7}$$

The constant in the equation above can be absorbed in the redefinition of the zero of the proper time. We can also, see from this equation that for r→ −∞, the propers time $\tau \to -\infty$.

Now we can rewrite Eq's (4.2) and (4.3) in terms of the affine parameter, the proper time of the particle.

$$R_{0101} = -\frac{4}{9\tau^2} \quad , \tag{4.8}$$

and

$$R_{0202} = R_{0303} = \frac{2}{9\tau^2} \quad , \tag{4.9}$$

The classical solutions for Eqs.(4.4) and (4.5), in terms of the affine parameter τ are given by

$$Z_1(\tau) = c_i \tau^{4/3} \tag{4.10}$$

and

$$Z_j(\tau) = c_j \tau^{2/3}, \tag{4.11}$$

where c_i and c_j, for $j = 2$ and 3, are constants. One can see that the expressions above are well defined at $r \leq 2M$.

After this brief analysis, let us return to the construction of the Schrödinger wave function by means of the invariant method. In order to do that we need to solve first the auxiliary equations

$$\frac{d^2\sigma_1}{d\tau^2} - \frac{4}{9\tau^2}\sigma_1 = \frac{K_1}{m^2\sigma_i^3} \tag{4.12}$$

and

$$\frac{d^2\sigma_j}{d\tau^2} + \frac{2}{9\tau^2}\sigma_j = \frac{K_j}{m^2\sigma_i^3} \quad , \tag{4.13}$$

where $j = 2$ and 3.

For the case $K_i > 0$, we obtained the following solutions:

$$\sigma_1(\tau) = \left[(1 + a_1^2)c_1^2 \mid \tau \mid^{8/3} + \frac{6a_1 K_1^{1/2} \mid \tau \mid}{5} + \frac{9K_1}{25c_1^2 \mid \tau \mid^{2/3}} \right]^{1/2}, \tag{4.14}$$

and

$$\sigma_j(\tau) = \left[(1 + a_j^2)c_j^2 \mid \tau \mid^{4/3} + 6a_j K_j^{1/2} \mid \tau \mid + 9K_j \mid \tau \mid^{2/3} \right]^{1/2}, \tag{4.15}$$

where a_1 and a_j are constants of integration.

For the case $K_i \leq 0$, we obtained

$$\sigma_1(\tau) = i \left[\mid b_1 \mid c_1 \mid \tau \mid^{8/3} + \frac{6 \mid K_1 \mid^{1/2}}{5} \mid \tau \mid \right]^{1/2}, \qquad (4.16)$$

and

$$\sigma_j(\tau) = i \left[\mid b_j \mid c_j \mid \tau \mid^{4/3} + 6 \mid K_j \mid^{1/2} \mid \tau \mid \right]^{1/2}, \qquad (4.17)$$

again b_1 and b_j are constants of integration.

Our next step in the construction of the wave function is to obtain the phase factor and the unitary transformation, and then to solve the eigenvalue equation (3.13). For the first two factors the knowledge of $\sigma_i(\tau)$ is essential. The phase factor is given by

$$\alpha_{\lambda_i}(\tau) = \alpha_{\lambda_i}^{(0)} \int \frac{d\tau}{\sigma_i^2}(\tau), \qquad (4.18)$$

$\alpha_{\lambda_i}^{(0)}$ being a constant, and the unitary operator is given by Eq.(3.10).

Finally in order to solve (3.13) we have to consider the two cases $K_i > 0$ and $K_i \leq 0$. For the first case, Eq. (3.13) becomes equivalent to an harmonic oscilator, with a discrete set of eigenvalues λ_n. For negative and vanishing values of K_i, Eq. (3.13) becomes equivalent to an inverted harmonic oscilator and a free system, respectively. Let us first analyse the discrete case:

i) For $K_i > 0$, the solutions for $\sigma_i(\tau)$ are given by Eq. (4.12) and (4.13), and the wave functions are

$$\Psi_{\lambda_i}(Z_i, \tau) = \frac{\exp^{i\alpha_{\lambda_i}(\tau)}}{\left[\pi^{1/2} \hbar^{1/2} n_i! 2^{n/2} \right]} \frac{exp\left[-\frac{m}{2i\hbar} \frac{\dot{\sigma}_i}{\sigma_i} Z_i^2 \right]}{\sigma_i^{1/2}} exp\left[-\frac{Z_i^2}{2\hbar \sigma_i^2} \right].$$

$$H_{\lambda_i} \left[\frac{Z_i}{\hbar^{1/2} \sigma_i} \right], \qquad (4.19)$$

where the phase $\alpha_{\lambda_i}(\tau)$ has been given above and H_{λ_i} are the Hermite polynomials.

ii) For $K_i < 0$ we also have the solutions for $\sigma_{\lambda_i}(\tau)$ given by Eq's (4.14) and (4.15), and the respective wave function are

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$$\Psi_{\lambda_i}(Z_i, \tau) = \exp^{i\alpha_{\lambda_i}(\tau)} \frac{exp\left[-\frac{m}{2i\hbar}\frac{\dot{\sigma_i}}{\sigma_i}Z_i^2\right]}{\sigma_i^{1/2}} D_{\nu_i}\left[\frac{(1+i)Z_i}{\hbar^{1/2}\sigma_i}\right], \qquad (4.20)$$

where $2\nu_i = -1 - i\lambda_i/\hbar$, λ_i being a continuos parameter, and $D_\nu(Y)$ are the parabolic cylinder functions [13].

iii) Finally for $K_i = 0$ the soluctions for $\sigma_i(\tau)$ are:

$$\sigma_1(\tau) = c_1 \mid \tau \mid^{4/3}, \qquad (4.21)$$

and

$$\sigma_2(\tau) = c_j \mid \tau \mid^{2/3} . \qquad (4.22)$$

The wave function are now given by:

$$\Psi_{\lambda_i}(Z_i, \tau) = \frac{exp\left[i(\frac{\dot{\sigma}}{2\hbar\sigma_i}Z_i^2 + \alpha_{\lambda i}(\tau) + \frac{\lambda_i Z_i}{\sigma_i})\right]}{\sigma_i^{1/2}} \qquad (4.23)$$

So the complete Schrödinger wave function which describes the radial motion of a test particle on the Schwarzschild space-time, is a linear combination of the solutions (4.19), (4.20) and (4.23).

5 Concluding Remarks

In this paper we have presented a new approach, a geometric one, to analyse, under a quantum point of view, the motion of a point particle interacting with an external background gravitational field. As a direct application of this formalism, we have investigated the motion of a point particle in the Schwarzschild black hole, and for this case we have obtained the relativistic Schrödinger wave function, using the invariant method. We would like to emphasize that, because our system is not conservative, there is no defined energy for the particle, and the probability density depends explicitly on the proper time τ.

This geometric approach can also be applied to other kinds of manifold, actually under investigation by us, and that some of its advantages are very clear like, for example, (i) it avoids the ordering problem for differential operators which generally appears in the Schrödinger equation defined on a

curved space-time [5], and (ii) it also avoids the horizon problem at r= 2M for the Schwarzschild space-time because our wave function is well defined on this region.

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